Barker College

Barker Student Number:________________

2006
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 1

Staff Involved:

- GDH*
- WMD*
- JM
- BTP
- GIC
- LJP
- CFR

65 copies

PM FRIDAY 11 AUGUST

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages
- Board-approved calculators may be used
- A table of standard integrals is provided on page 9
- ALL necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged working

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

Marks

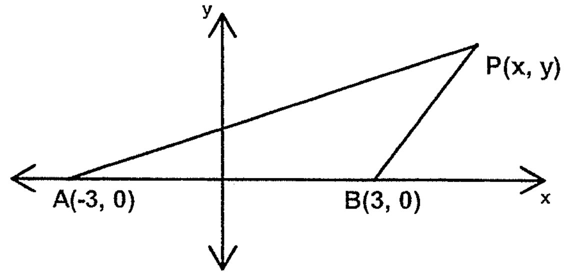
Question 1 (1[illegible]rks) **[BEGIN A NEW PAGE]**

(a) Point A has coordinates (-2, 4). Point B has coordinates (10, -8). Find the coordinates of the point P that divides the interval AB **externally** in the ratio $3:2$. 2

(b) Find $\int x\sqrt{2x-1}\,dx$ using the substitution $u = 2x - 1$. 3

(c) State whether the following claim is true or false and give a reason why:
"Because there are two types of students at Barker (day and boarder), the probability that a randomly selected Barker student is a boarder is 50%." 1

(d)



The locus of P follows this rule:
"The gradient of PA is one unit less than the gradient of PB."

(i) Show that this locus has equation: $x^2 = 6y + 9$. 2

(ii) Hence, or otherwise, find the coordinates of the focus of this locus. 1

(e) Solve: $\dfrac{1}{x^3} > \dfrac{1}{x^5}$. 3

Question 2 (12 marks) [BEGIN A NEW PAGE]

Marks

(a) By writing $\sin(-15°)$ in the form $\sin(A - B)$, find the exact value of $\sin(-15°)$. 2

(b) Consider the functions $f(x) = \cos^{-1}(2x)$ and $g(x) = \sin^{-1} x$.

(i) Sketch $f(x) = \cos^{-1}(2x)$. 2

(ii) Prove that the x-coordinate of the point of intersection of $f(x)$ and $g(x)$ is $\dfrac{1}{\sqrt{5}}$. 2

(iii) Show that the gradients of the tangents to $f(x)$ and $g(x)$ at their point of intersection are $-2\sqrt{5}$ and $\dfrac{\sqrt{5}}{2}$ respectively. 3

(iv) Write the expansion of $\tan(\beta - \alpha)$. 1

(v) Hence, or otherwise, find the acute angle between $f(x)$ and $g(x)$ at their point of intersection (to the nearest degree). 2

Question 3 (1[illegible]arks) [BEGIN A NEW PAGE]

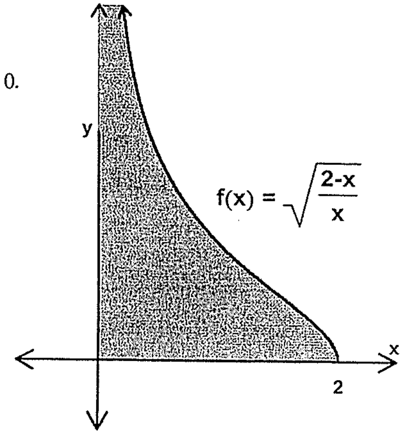
Marks

(a) (i) If $y = \ln(\sin x)$, find $\dfrac{dy}{dx}$. 1

(ii) Hence, or otherwise, find $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cot x \, dx$. 2

(b) Find $\displaystyle\int_0^{\frac{\pi}{12}} \cos^2(3x) \, dx$. 3

(c) The curve on the right has an asymptote at $x = 0$. The shaded area 'goes forever' but its value has a limit. Steffi Graph was trying to find this limit but was unable to integrate $f(x)$. Her friend Monica suggested she use the inverse function to find the shaded area.



(i) State the domain and range of the inverse function $f^{-1}(x)$. 1

(ii) Roughly sketch the inverse function $f^{-1}(x)$. 1

(iii) Show that $f^{-1}(x) = \dfrac{2}{1 + x^2}$. 2

(iv) Hence, or otherwise, find the limit that the value of the shaded area approaches. 2

Question 4 (12 marks) **[BEGIN A NEW PAGE]** **Marks**

(a) Consider the curve: $y = \dfrac{x^2 - 3}{x + 2}$

(i) Find all intercepts and the equation of the vertical asymptote. **2**

(ii) Find and determine the nature of the stationary points. **3**

(iii) Show that $(x - 2) + \dfrac{1}{x + 2} = \dfrac{x^2 - 3}{x + 2}$. **1**

(iv) Hence, or otherwise, find the equation of any non-vertical asymptotes by considering what happens as $x \to \pm\infty$. **1**

(v) Sketch the curve showing all the above features (you can assume there are no points of inflexion). **1**

(b) Prove, by mathematical induction, that $5^n + 3$ is divisible by 2 for $n \geq 0$ where n is an integer. **4**

Question 5 ([illegible] marks) **[BEGIN A NEW PAGE]** **Marks**

(a) A particle moves in a straight line and its position in metres at any time t seconds is given by the equation: $x = 5\cos(2t) - 12\sin(2t)$.

(i) Show, by differentiation, that the motion is simple harmonic. **2**

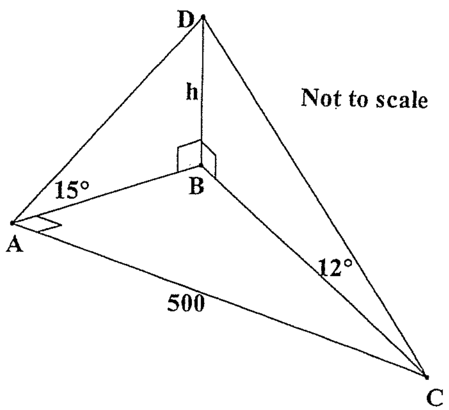
(ii) State the period of the motion. **1**

(iii) Express x in the form $R\cos(2t + \alpha)$. **2**

(iv) Hence, or otherwise, state the amplitude of the motion. **1**

(v) Find the general solution for all the times when the particle is at the centre of the motion. **2**

(b) From a point A, the top of a tower BD directly north of A has an angle of elevation of $15°$. After walking 500 metres on a bearing of $90°$, the top of the tower has an angle of elevation of $12°$.
Let h be the height of the tower.



(i) Give an expression for AB in terms of h. **1**

(ii) Hence, find the height of the tower (to the nearest metre). **3**

Question 6 (12 marks) [BEGIN A NEW PAGE] **Marks**

(a) Newton's Law of Cooling can be written with t as the subject as follows:

$$t = -\frac{1}{A}\ln\left(\frac{B - C}{D}\right).$$

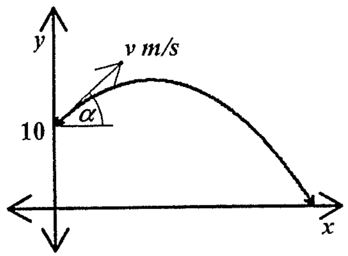
Showing steps of working, make B the subject of this formula. **2**

(b) The length of each edge of a cube is x cm.

(i) Write expressions for the surface area (A) and volume (V) of the cube and hence find $\frac{dA}{dx}$ and $\frac{dV}{dx}$. **1**

(ii) The surface area of the cube increases at a rate of 6 cm^2/second. Find the rate of change of the volume of the cube when the length of each edge is 5 cm. **2**

(c) A stone is thrown from a 10m high cliff with velocity v m/s at an angle of projection α. The stone's horizontal displacement from the origin, t seconds after being thrown, is given by the equation $x = vt\cos\alpha$. Do not prove this.



(i) Given that $\ddot{y} = -g$, prove that the stone's vertical displacement from the origin, t seconds after being thrown, is given by $y = vt\sin\alpha - \frac{gt^2}{2} + 10$. **3**

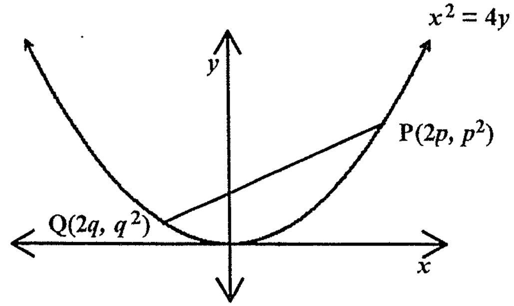
(ii) Show that $y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2v^2} + 10$. **1**

(iii) Given that $\alpha = 45°$, $g = 10 m/s^2$ and $v = 15 m/s$, how far from the base of the cliff does the stone land? **3**

Question 7 (arks) [BEGIN A NEW PAGE] **Marks**

(a) It is known that if $\frac{dy}{dx} = y$, then $y = e^x$ is a solution (since $\frac{dy}{dx} = e^x = y$).
If $\frac{dy}{dx} = \frac{1}{y}$, find a solution for y. **2**

(b)



(i) Show that the gradient of PQ is $\frac{p + q}{2}$. **1**

For the remainder of the question, assume that PQ is a focal chord, passing through the focus F(0, 1).

(ii) Show that $pq = -1$. **2**

(iii) Show that the equation of PQ is $y = \left(\frac{p^2 - 1}{2p}\right)x + 1$. **1**

(iv) Let A be the area bounded by the parabola and the focal chord.
Show that $A = \frac{1}{3}\left(p^3 + 3p + \frac{3}{p} + \frac{1}{p^3}\right)$.
[You may assume that $p > 0$ and $q < 0$] **3**

(v) Hence, or otherwise, find the value of p that gives the minimum area found in part (iv). **3**

End of Paper

BARKER COLLEGE

Mathematics Extension 1 2006 Trial Solutions

1 (a) $A(-2,4)$ $B(10,-8)$

$3 : -2$

$$P = \left(\frac{3\times 10 - 2\times(-2)}{3-2}, \frac{3\times(-8)-2\times 4}{3-2}\right) \checkmark$$

$$= \underline{\underline{(34, -32)}} \checkmark$$

1 mark for internal division with correct answer $(5\tfrac{1}{5}, -3\tfrac{1}{5})$

(b) $\int x\sqrt{2x-1}\,dx$

$u = 2x-1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$ & $x = \frac{u+1}{2}$ ✓

$$= \int \frac{(u+1)}{2}\cdot u^{\frac{1}{2}}\cdot\frac{du}{2}$$

$$= \frac{1}{4}\int (u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du \checkmark$$

$$= \frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right] + c$$

$$= \underline{\underline{\frac{1}{10}(2x-1)^{\frac{5}{2}} + \frac{1}{6}(2x-1)^{\frac{3}{2}} + c}} \checkmark$$

No penalty for not having a constant of integration

(c) The statement is false as there are far more day students than boarders which means you are more likely to randomly select a day student. ✓

Must justify answer to get the mark.

(d) (i) $m_{PA} = m_{PB} - 1$

$$\therefore \frac{y}{x+3} = \frac{y}{x-3} - 1 \checkmark$$

$$y(x-3) = y(x+3) - (x+3)(x-3)$$

$$\cancel{xy} - 3y = \cancel{xy} + 3y - (x^2-9) \checkmark$$

$$x^2 - 9 = 6y$$

$$\underline{x^2 = 6y + 9}$$

(ii) $x^2 = 4(1.5)(y+1.5)$

$\therefore$ vertex is $(0,-1.5)$ & focal length $= 1.5$ u

$\therefore$ Focus is the $\underline{\text{origin}}$ ✓

1 (e) $\frac{1}{x^3} > \frac{1}{x^5}$, $x \neq 0$

$$x^6 \times \frac{1}{x^3} > x^6 \times \frac{1}{x^5} \checkmark$$ ← can also multiply by x^8, x^{10}, etc.

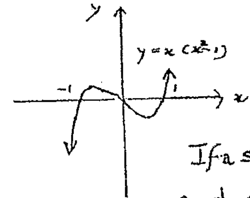
$$x^3 > x$$

$$x^3 - x > 0$$

$$x(x^2-1) > 0$$

$$x(x+1)(x-1) > 0 \checkmark$$

$$\therefore \underline{\underline{-1 < x < 0 \text{ or } x > 1}} \checkmark$$



Ignore any irregularities with $\le$ or $\ge$ instead of $<$ and $>$.

If a student multiplies thru' by x^5 and gets $x < -1$ or $x > 1$ award a total of 1 mark.

2 (a) $\sin(-15^\circ)$

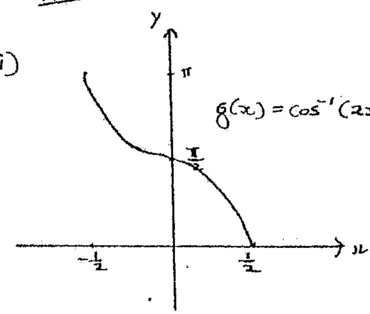
$$= \sin(30^\circ - 45^\circ)$$

$$= \sin 30^\circ \cos 45^\circ - \cos 30^\circ \sin 45^\circ \checkmark$$

$$= \frac{1}{2}\times\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$$ ✓ − ignore subsequent errors

$$= \underline{\underline{\frac{1-\sqrt{3}}{2\sqrt{2}}}}$$

(b) (i)



correct domain 1 mark

correct shape & range 1 mark

(ii) Let $f(x) = g(x)$

i.e. $\cos^{-1}(2x) = \sin^{-1}(x)$

let $\sin^{-1}x = \alpha$ $\Rightarrow x = \sin\alpha$

let $\cos^{-1}(2x) = \alpha$ $\Rightarrow 2x = \cos\alpha$ ✓

Now $\sin^2\alpha + \cos^2\alpha = 1$

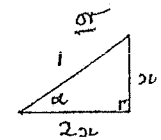
$$\therefore x^2 + (2x)^2 = 1 \checkmark$$

$$5x^2 = 1$$

$$x^2 = \frac{1}{5}$$

$x = \frac{1}{\sqrt{5}}$, since $x > 0$ − see graph ← ignore $\pm$ in marking.

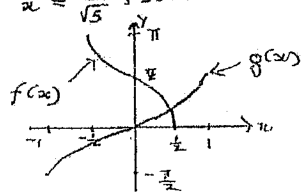
or



✓

$x^2 + (2x)^2 = 1$ ✓

etc.



(2) (c)

(iii) $f'(x) = \dfrac{-2}{\sqrt{1-(2x)^2}}$ ✓

$f'\left(\frac{1}{\sqrt{5}}\right) = \dfrac{-2}{\sqrt{1-\frac{4}{5}}}$

$= \dfrac{-2}{\frac{1}{\sqrt{5}}}$ ✓

$= -2\sqrt{5}$

$g'(x) = \dfrac{1}{\sqrt{1-x^2}}$

$g'\left(\frac{1}{\sqrt{5}}\right) = \dfrac{1}{\sqrt{1-\frac{1}{5}}}$

$= \dfrac{1}{\sqrt{\frac{4}{5}}}$

$= \dfrac{1}{\frac{2}{\sqrt{5}}}$ ✓

$= \dfrac{\sqrt{5}}{2}$

(iv) $\tan(\beta - \alpha) = \dfrac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha}$ ✓

(v) Let the required angle be ϕ

$\tan\phi = \left|\dfrac{m_1 - m_2}{1+m_1 m_2}\right|$

$= \left|\dfrac{\frac{\sqrt{5}}{2} + 2\sqrt{5}}{1-5}\right|$ ✓

$= \dfrac{5\sqrt{5}}{8}$

$\therefore \phi = 54.41\ldots^\circ$

$= 54^\circ$ (n.deg.) ✓ Don't worry about accuracy.

(3) (a) (i) $y = \ln(\sin x)$

$\dfrac{dy}{dx} = \dfrac{\cos x}{\sin x}$ ✓

$= \cot x$

(ii) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cot x \, dx$

$= \Big[\ln(\sin x)\Big]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$ ✓

$= \ln\left(\sin\frac{3\pi}{4}\right) - \ln\left(\sin\frac{\pi}{4}\right)$

$= \ln\left(\frac{1}{\sqrt{2}}\right) - \ln\left(\frac{1}{\sqrt{2}}\right)$ ✓

$= 0$

(b) $\displaystyle\int_0^{\frac{\pi}{12}} \cos^2(3x)\, dx$

$= \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{12}} (\cos 6x + 1)\, dx$ ✓

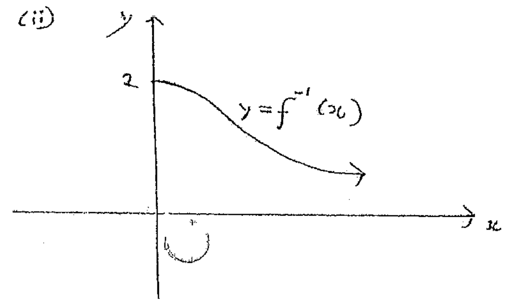
$= \dfrac{1}{2}\left[\dfrac{\sin 6x}{6} + x\right]_0^{\frac{\pi}{12}}$ ✓

$= \dfrac{1}{2}\left\{\left(\frac{1}{6} + \frac{\pi}{12}\right) - (0+0)\right\}$ ✓

$= \dfrac{2+\pi}{24}$

(c) (i) $\left.\begin{array}{l} D_{f^{-1}} : x \geq 0 \\ R_{f^{-1}} : 0 < y \leq 2 \end{array}\right\} 1$ Must have inequality signs exactly right for the mark.

(ii)



y-intercept and shape required for 1 mark.

③ (c)

(iii)

$$y = f(x) = \sqrt{\frac{2-x}{x}}$$

$\therefore \quad x = \sqrt{\frac{2-y}{y}}$ is inverse function ✓

$$x^2 = \frac{2-y}{y}$$

$$x^2 = \frac{2}{y} - 1 \quad ✓$$

$$x^2 + 1 = \frac{2}{y}$$

$$y = \frac{2}{x^2+1} = f^{-1}(x)$$

or $\rightarrow$

$$x^2 y = 2 - y$$

$$x^2 y + y = 2$$

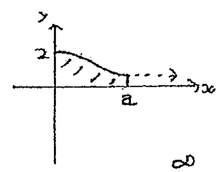
$$y(x^2+1) = 2 \quad ✓$$

(iv) Consider

$$\int_0^a \frac{2}{1+x^2}\,dx = 2\left[\tan^{-1}x\right]_0^a \quad ✓$$

$$= 2\{\tan^{-1}a - \tan^{-1}0\}$$

$$= 2\tan^{-1}a$$



$$\text{Req'd area} = \lim_{a\to\infty} 2\tan^{-1}a$$

$$= 2\lim_{a\to\infty}\tan^{-1}a$$

$$= 2\times\frac{\pi}{2} \quad ✓$$

$$= \pi \text{ units}^2$$

OR

$$A = \int_0^\infty \frac{2}{1+x^2}\,dx \quad ✓$$

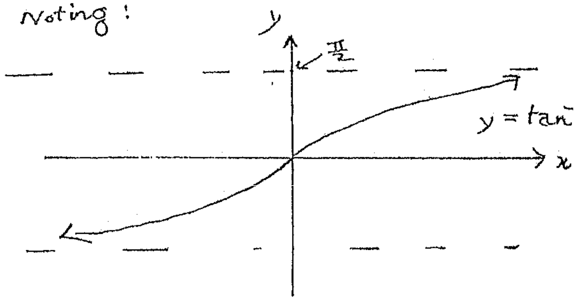
$$= 2\left[\tan^{-1}x\right]_0^\infty$$

$$= 2\{\tan^{-1}\infty - \tan^{-1}0\}$$

$$= 2\left\{\frac{\pi}{2} - 0\right\} \quad ✓$$

$$= \pi \text{ units}^2$$

Noting:



④ (a)

(i)

$$y = \frac{(x-\sqrt{3})(x+\sqrt{3})}{x+2}$$

$\Rightarrow x = -2$ is vertical asymptote

when $x = 0$, $y = -\frac{3}{2}$ [y-intercept] } 1 mark

when $y = 0$, $x = \pm\sqrt{3}$ [x-intercepts] ✓

(ii)

$$\frac{dy}{dx} = \frac{(x+2)2x - (x^2-3)1}{(x+2)^2} \quad ✓$$

$$= \frac{2x^2 + 4x - x^2 + 3}{(x+2)^2}$$

$$= \frac{x^2 + 4x + 3}{(x+2)^2}$$

$$= \frac{(x+1)(x+3)}{(x+2)^2}$$

$\Rightarrow$ stat. pts at $x = -1, -3$ ✓

when $x = -1$, $y = \frac{1-3}{1} = -2$

when $x = -3$, $y = \frac{9-3}{-1} = -6$

test $(-1,-2)$:

x	$-1\frac{1}{2}$	-1	0
$\frac{dy}{dx}$	$\frac{(-\frac{1}{2})(1\frac{1}{2})}{(\frac{1}{2})^2}$ neg.	0	$\frac{3}{4}$ pos.

min. t.p.

test $(-3,-6)$:

x	-4	-3	$-2\frac{1}{2}$
$\frac{dy}{dx}$	$\frac{(-3)(-1)}{(-2)^2}$ pos.	0	$\frac{(-1\frac{1}{2})(\frac{1}{2})}{(-\frac{1}{2})^2}$ neg.

max. t.p.

} 1 mark

(iii)

$$(x-2) + \frac{1}{x+2} = \frac{(x-2)(x+2)+1}{x+2}$$

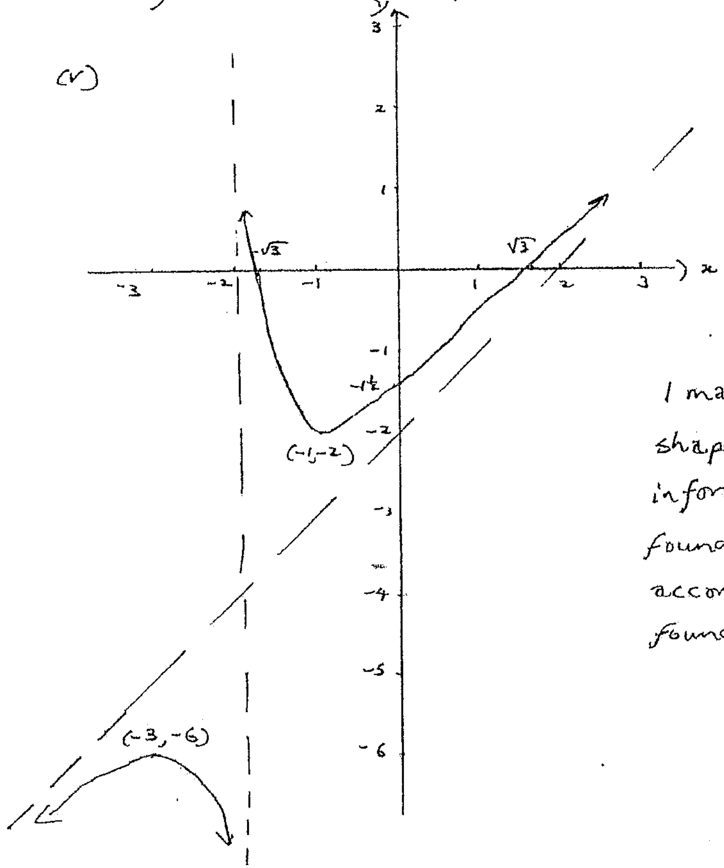
$$= \frac{x^2 - 4 + 1}{x+2} \quad ✓$$

$$= \frac{x^2-3}{x+2}$$

4 (a)(iv) as $x \to \infty$, $(x-2) + \frac{1}{x+2} \to x-2$

$\therefore$ $y = x-2$ is an oblique asymptote. ✓

(v)



1 mark for correct shape - (all other information has been found earlier) - according to values found above.

b) when $n=0$, $5^0+3 = 4|2$

$\Rightarrow$ $5^n+3|2$ for $n=0$. ✓

Assume $5^k+3 = 2J$, $J \in \mathbb{Z}^+$

i.e. $5^k = 2J-3$

Required to prove $5^{k+1}+3|2$

Now $5^{k+1}+3 = 5.5^k+3$ ✓

$= 5(2J-3)+3$ by inductive supposition

$= 10J-12$

4 (b) (cont.)

$\Rightarrow 5^{k+1}+3|2$ if $5^k+3|2$ (*)

Since $5^n+3|2$ for $n=0$ it is true for $n=1$ and hence true for all subsequent positive integral values of n by (*). ✓

5 (a)(i) $x = 5\cos(2t) - 12\sin(2t)$

$\dot{x} = -10\sin(2t) - 24\cos(2t)$ ✓

$\ddot{x} = -20\cos(2t) + 48\sin(2t)$

$= -4(5\cos(2t) - 12\sin(2t))$

$= -4x$ ✓

which is in the form $\ddot{x} = -n^2 x$.

Hence the motion is simple harmonic.

(ii) Period $= \frac{2\pi}{2} = \underline{\pi}$ seconds ✓

(iii) $R\cos(2t+\alpha)$

$= R\{\cos(2t)\cos\alpha - \sin(2t)\sin\alpha\}$

$\therefore$ Let $5\cos(2t) - 12\sin(2t) \equiv R\cos(2t)\cos\alpha - R\sin(2t)\sin\alpha$

$\Rightarrow$ $R\cos\alpha = 5$ & $R\sin\alpha = 12$ ✓

$R^2\cos^2\alpha + R^2\sin^2\alpha = 5^2+12^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 169$

$R = 13$ $[R>0]$

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{12}{5} \Rightarrow \tan\alpha = \frac{12}{5}$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{12}{5}\right)$ [α acute]

1 mark for getting both R and α.

$\therefore$ $x = \underline{13\cos\left(2t + \tan^{-1}\left(\frac{12}{5}\right)\right)}$

also $\alpha = \sin^{-1}\left(\frac{12}{13}\right)$

$= \cos^{-1}\left(\frac{5}{13}\right)$

5 (a)

(iv) Amplitude = 13 metres.

(v) Particle is at centre of motion when acceleration is zero at $x = 0$ as eq'n is in form $\ddot{x} = -n^2 x$

i.e. when $5\cos(2t) - 12\sin(2t) = 0$

i.e. when $13\cos\left(2t + \tan^{-1}\left(\frac{12}{5}\right)\right) = 0$

$\Rightarrow \cos\left(2t + \tan^{-1}\left(\frac{12}{5}\right)\right) = 0$

$2t + \tan^{-1}\left(\frac{12}{5}\right) = 2n\pi \pm \cos^{-1}(0)$

$2t + \tan^{-1}\left(\frac{12}{5}\right) = 2n\pi \pm \frac{\pi}{2}$

$$t = \frac{2n\pi + \frac{\pi}{2} - \tan^{-1}\left(\frac{12}{5}\right)}{2}, \quad \frac{2n\pi - \frac{\pi}{2} - \tan^{-1}\left(\frac{12}{5}\right)}{2}$$

choosing n to be an integer such that $t > 0$.

(b)(i) $\angle ADB = 75^\circ$ ($\angle$ sum Δ)

$\therefore \frac{AB}{h} = \tan 75^\circ$ or $\frac{h}{AB} = \tan 15^\circ$, $AB = \frac{h}{\tan 15^\circ} = h\cot 15^\circ$ or $AB = \frac{h}{\tan 15^\circ}$

$\therefore AB = h\tan 75^\circ$

(ii) Similarly $BC = h\tan 78^\circ$

By Pythagoras: $AB^2 + AC^2 = BC^2$

$\therefore h^2\tan^2 75^\circ + 500^2 = h^2\tan^2 78^\circ$

$h^2(\tan^2 78^\circ - \tan^2 75^\circ) = 250\,000$

$$h = \frac{500}{\sqrt{\tan^2 78^\circ - \tan^2 75^\circ}}$$

$= 174.55\ldots$

$= 175$ m (nearest metre)

No penalty for accuracy/units.

6 (a) $t = -\frac{1}{A}\ln\left(\frac{B-C}{D}\right)$

$-At = \ln\left(\frac{B-C}{D}\right)$

$e^{-At} = \frac{B-C}{D}$

$B - C = De^{-At}$

$\therefore B = C + De^{-At}$

(b) (i) $A = 6x^2 \Rightarrow \frac{dA}{dx} = 12x$

$V = x^3 \Rightarrow \frac{dV}{dx} = 3x^2$

1 mark

If any of this done in (ii) give mark.

(ii) $\frac{dA}{dt} = 6\text{ cm}^2/\text{s}$

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

when $x = 5$, $6 = (12 \times 5) \times \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{1}{10}$ cm/s

and $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$= (3 \times 5^2) \times \frac{1}{10}$

$= 7.5\text{ cm}^3/\text{s}$

or $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dA} \times \frac{dA}{dt}$

$= 3x^2 \times \frac{1}{12x} \times 6$

when $x = 5$, $\frac{dV}{dt} = 3 \times 5^2 \times \frac{1}{12 \times 5} \times 6$

$= 7.5\text{ cm}^3/\text{s}$

OR $\frac{dx}{dt} = \frac{dx}{dA} \times \frac{dA}{dt}$

$= \frac{1}{12x} \times 6$

$= \frac{1}{2x}$

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$= 3x^2 \times \frac{1}{2x}$

$= \frac{3x}{2}$

when $x = 5$, $\frac{dV}{dt} = \frac{3 \times 5}{2}$

$= 7.5\text{ cm}^3/\text{s}$

(c) $\ddot{y} = -g$

$\dot{y} = \int -g\,dt$

$= -gt + c_1$

when $t = 0$, $\dot{y} = V\sin\alpha$:

$V\sin\alpha = c_1$

$\therefore \dot{y} = -gt + V\sin\alpha$

$y = \int(-gt + V\sin\alpha)\,dt$

$= -\frac{gt^2}{2} + Vt\sin\alpha + c_2$

when $t = 0$, $y = 10$:

$10 = c_2$

$\therefore y = Vt\sin\alpha - \frac{gt^2}{2} + 10$

Students must demonstrate by substitution how they arrive at both the constants of integration - do not award full marks to solutions that skip any key steps in the process.

6 (c) (ii)

$x = vt\cos\alpha \Rightarrow t = \frac{x}{v\cos\alpha}$

$\therefore y = v \times \frac{x}{v\cos\alpha} \times \sin\alpha - \frac{g \times \left(\frac{x}{v\cos\alpha}\right)^2}{2} + 10$ ✓

$= x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha} + 10$

$= x\tan\alpha - \frac{gx^2\sec^2\alpha}{2v^2} + 10$

(iii) $\alpha = 45°,\ g = 10\text{ m/s}^2,\ v = 15\text{ m/s}$

$\Rightarrow y = x - \frac{10x^2 \times \cancel{2}}{\cancel{2} \times 225} + 10$ ✓ {substituting & calculating the value of $\sec^2 45°$}

When $y = 0$, $0 = x - \frac{10x^2}{225} + 10$

$\frac{\cancel{10}^{2}x^2}{\cancel{225}_{45}} - x - 10 = 0$

$2x^2 - 45x - 450 = 0$ ✓ ← Accept any correct eq'n in general quadratic form.

$(2x + 15)(x - 30) = 0$ or $x = \frac{45 \pm \sqrt{5625}}{4}$

$\therefore x = 30 \quad (x > 0)$ ✓ ← Must exclude other solution to quadratic.

So stone lands 30 m from base of cliff.

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7 (a) $\frac{dx}{dy} = y$ if $\frac{dy}{dx} = \frac{1}{y}$

$\therefore \int \frac{dx}{dy}\,dy = \int y\,dy$ ✓

$x = \frac{y^2}{2} + \text{a constant}$

$y^2 = 2x + \text{a constant}$

$\therefore y = \pm\sqrt{2x} + c$ ✓ → accept $y = \sqrt{2x}$ or $y = -\sqrt{2x}$ {with or without a constant}

(b) (i) $m_{PQ} = \frac{p^2 - q^2}{2p - 2q}$

$= \frac{(p+q)(p-q)}{2(p-q)}$ ✓

$= \frac{p+q}{2}$

7 (b)

(ii) Since PQ passes through F,

$m_{PF} = m_{PQ}$

$\therefore \frac{p^2 - 1}{2p} = \frac{p+q}{2}$ ✓

$\cancel{2p^2} - 2 = \cancel{2p^2} + 2pq$ ✓

$\therefore pq = -1$

(iii) As above $m_{PQ} = m_{PF}$

$= \frac{p^2-1}{2p}$

Also PQ passes through $(0, 1)$ so its y-intercept is 1. — both for 1 mark

Hence using $y = mx + b$,

PQ is $y = \left(\frac{p^2-1}{2p}\right)x + 1$.

OR $y - 1 = \left(\frac{p+q}{2}\right)(x - 0)$

$y = \left(\frac{p+q}{2}\right)x + 1$

$= \left(\frac{p - \frac{1}{p}}{2}\right)x + 1$ ✓

$= \left(\frac{p^2-1}{2p}\right)x + 1$

(iv) $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

& $pq = -1 \Rightarrow q = -\frac{1}{p}$

$\therefore A = \int_{-\frac{2}{p}}^{2p} \left\{ \left(\frac{p^2-1}{2p}\right)x + 1 - \frac{x^2}{4} \right\} dx$ ✓

$= \left[\left(\frac{p^2-1}{4p}\right)x^2 + x - \frac{x^3}{12} \right]_{-\frac{2}{p}}^{2p}$

$= \left\{ \left(\frac{p^2-1}{\cancel{4p}}\right)\cancel{4}p^{\cancel{2}} + 2p - \frac{8p^3}{12} \right\} - \left\{ \left(\frac{p^2-1}{\cancel{4}p}\right) \times \frac{\cancel{4}}{p^2} - \frac{2}{p} + \frac{8}{12p^3} \right\}$ ✓

$= p^3 - p + 2p - \frac{2p^3}{3} - \frac{1}{p} + \frac{1}{p^3} + \frac{2}{p} - \frac{2}{3p^3}$

$= \frac{p^3}{3} + p + \frac{1}{p} + \frac{1}{3p^3}$ ✓

$= \frac{1}{3}\left(p^3 + 3p + \frac{3}{p} + \frac{1}{p^3}\right)$

7 (b)
(iv) (cont.)

Alternatively:

$$A = \int_{2q}^{2p} \left\{ \frac{(p^2-1)}{2p}x + 1 - \frac{x^2}{4} \right\} dx$$

$$= \left[\left(\frac{p^2-1}{4p}\right)x^2 + x - \frac{x^3}{12} \right]_{2q}^{2p}$$

$$= \left\{ \left(\frac{p^2-1}{4p}\right) \times 4p^2 + 2p - \frac{8p^3}{12} \right\} - \left\{ \frac{(p^2-1)}{4p} \times 4q^2 + 2q - \frac{8q^3}{12} \right\}$$

$$= p^3 - p + 2p - \frac{2p^3}{3} - pq^2 + \frac{q^2}{p} - 2q + \frac{2q^3}{3}$$

$$= \frac{p^3}{3} + p - p \times \left(-\frac{1}{p}\right)^2 + \frac{\left(-\frac{1}{p}\right)^2}{p} - 2\left(-\frac{1}{p}\right) + \frac{2\left(-\frac{1}{p}\right)^3}{3} \quad \left\{ \text{since } q = -\frac{1}{p} \right\}$$

$$= \frac{p^3}{3} + p - \frac{1}{p} + \frac{1}{p^3} + \frac{2}{p} - \frac{2}{3p^3}$$

$$= \frac{p^3}{3} + p + \frac{1}{p} + \frac{1}{3p^3}$$

$$= \frac{1}{3}\left(p^3 + 3p + \frac{3}{p} + \frac{1}{p^3}\right)$$

(v) $A = \frac{1}{3}\left(p^3 + 3p + 3p^{-1} + p^{-3}\right)$

$$\frac{dA}{dp} = \frac{1}{3}\left(3p^2 + 3 - 3p^{-2} - 3p^{-4}\right)$$

$$= p^2 + 1 - \frac{1}{p^2} - \frac{1}{p^4}$$

when $\frac{dA}{dp} = 0$, $p^2 + 1 - \frac{1}{p^2} - \frac{1}{p^4} = 0$

$\therefore p^6 + p^4 - p^2 - 1 = 0$

$p^4(p^2+1) - 1(p^2+1) = 0$

$(p^4-1)(p^2+1) = 0$

$\therefore p^4 - 1 = 0$ {no real sol'ns for $p^2+1=0$}

$\therefore p^4 = 1 \Rightarrow p = 1 \quad (p>0)$

when $p = \frac{1}{2}$, $\frac{dA}{dp} = \left(\frac{1}{2}\right)^2 + 1 - \frac{1}{\left(\frac{1}{2}\right)^2} - \frac{1}{\left(\frac{1}{2}\right)^4}$

$= \frac{1}{4} + 1 - 4 - 16 < 0$

when $p = 2$, $\frac{dA}{dp} = 4 + 1 - \frac{1}{4} - \frac{1}{16} > 0$

min. t.p.